Pathwise stochastic integration with finite variation processes uniformly approximating càdlàg processes

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Abstract

For any real-valued stochastic process X with càdlàg paths we define non-empty family of processes, which have locally finite total variation, have jumps of the same order as the process X and uniformly approximate its paths on compacts. This allows to decompose any real-valued stochastic process with càdlàg paths and infinite total variation into a sum of uniformly close on compacts, finite variation process and an adapted process, with arbitrary small amplitude but infinite total variation. Application of the defined class is the definition of the stochastic integral with respect to the process X as a limit of pathwise Lebesgue-Stieltjes integrals. This construction leads to the stochastic integral with some correction term.

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1. Introduction

Let $X = (X_t)_{t\geq 0}$ be a real-valued stochastic process with càdlàg paths and let $0 \leq a < b$. The total variation of the process X on the interval $[a; b]$ is defined with the following formula

$$TV(X, [a; b]) = \sup_n \sup_{a\leq t_0<t_1<\ldots<t_n\leq b} \sum_{i=1}^{n} \left|X_{t_i} - X_{t_{i-1}}\right|.$$

Unfortunately, many of the most important families of stochastic processes are characterized with the "wild" behaviour, demonstrated by their infinite total variation. This fact arguably caused the need of the development of the general theory of stochastic integral. The main idea allowing to overcome the problematic infinite total variation and define stochastic integral with respect to a semimartingale utilizes the fact that the quadratic variation of the semimartingale is still finite. The approach used in this article is somewhat different. It is similar to the old approach of Wong and Zakai [9] and is based on the simple observation that in the neighborhood (in sup norm) of every càdlàg function defined on some compact interval one easily finds another function with finite total variation. Thus, for every $c > 0$, the process X may be decomposed as the sum

$$X = X^c + (X - X^c)$$

where X^c is a "nice" process with finite total variation and the difference $X - X^c$ is a process with small amplitude but possibly "wild" behavior with infinite total variation. More precisely, let F be some fixed, right continuous filtration such that X is adapted to F. Now, for every $c > 0$ we introduce (non-empty, as it will be shown in the sequel) family $\mathcal{X}^c$ of processes with càdlàg paths, satisfying the following conditions. If $X^c \in \mathcal{X}^c$ then

1. the process X^c has locally finite total variation;
2. X^c has càdlàg paths;
3. for every $T \geq 0$ there exists such $K_T < +\infty$ that for every $t \in [0;T]$, $|X_t - X_t^c| \leq K_T c$;
4. for every $T \geq 0$ there exists such $L_T < +\infty$ that for every $t \in [0;T]$, $|\Delta X_t^c| \leq L_T |\Delta X_t|$;
5. the process X^c is adapted to the filtration F.

We will prove that if processes X and Y are càdlàg semimartingales on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, F)$, with a probability measure $\mathbb{P}$, such that usual hypotheses hold (cf. [8, Sect. 1.1]), then the sequence of pathwise Lebesgue-Stieltjes integrals

$$\int_0^T Y_- \mathrm{d}X^c,\ c > 0,$$

with $X^c \in \mathcal{X}^c$, tends uniformly in probability $\mathbb{P}$ on compacts as $c \downarrow 0$ to $\int_0^T Y_- \mathrm{d}X + [X^{cont}, Y^{cont}]_T$; $\int_0^T Y_- \mathrm{d}X$ denotes here the (semimartingale) stochastic integral and X^{cont} and Y^{cont} denote continuous parts of X and Y respectively. Moreover, for any square summable sequence $(c(n))_{n\geq 1}$ we get $\mathbb{P}$ a.s. and uniform on compacts convergence of the sequence $\int_0^T Y_- \mathrm{d}X^{c(n)}, n = 1, 2, ...$ (cf. Theorem 6). We shall stress here that for every $c > 0$ and each pair of càdlàg paths $(X(\omega), Y(\omega)), \omega \in \Omega$, the value of $\int_0^T Y_-(\omega) \mathrm{d}X^c(\omega)$ (and thus the limit, if it exists) is independent of the probability measure $\mathbb{P}$. Thus we obtain a result in the spirit of Bichteler, see [1], [5], and the recent result of Nutz [7], where operations leading to the stochastic integral, independent of probability measures and filtrations are considered. Our approach seems to be simpler and more natural, in the sense that the introduction of

the family $\mathcal{X}^c$ seems to be quite natural. However, we need to impose a stronger condition on the integrand - that it is also a semimartingale and in the limit we do not obtain the Itô integral.

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2. Existence of the sequence $(X^c)_{c>0}$

In this section we will prove that for every $c > 0$ the family of processes $\mathcal{X}^c$, satisfying the conditions (1)-(5) of Section 1 is non-empty. For given $c > 0$ we will simply construct a process X^c satisfying all these conditions. We start with few definitions.

For fixed $c > 0$ we define two stopping times

$$T_u^{2c}X = \inf\left\{s \geq 0: \sup_{t\in[0;s]} X_t - X_0 > c\right\},$$
$$T_d^{2c}X = \inf\left\{s \geq 0: X_0 - \inf_{t\in[0;s]} X_t > c\right\}.$$

Assume that $T_d^{2c}X \geq T_u^{2c}X$, i.e. the first upward jump of the process X from X_0 of size c appears before the first downward jump of the same size c or both times are infinite (there is no upward or downward jump of size c). Note that in the case $T_d^{2c}X < T_u^{2c}X$ we may simply consider the process $-X$. Now we define sequences $\left(T_{d,k}^{2c}\right)_{k=1}^{\infty}$, $\left(T_{u,k}^{2c}\right)_{k=1}^{\infty}$ in the following way: $T_{u,0}^{2c} = T_u^{2c}X$

and for $k = 0, 1, 2, ...$

$$T_{d,k}^{2c} = \begin{cases} \inf\left\{s \geq T_{u,k}^{2c} : \sup_{t \in \left[T_{u,k}^{2c}; s\right]} X_t - X_s > 2c\right\} & \text{if } T_{u,k}^{2c} < +\infty, \\ +\infty & \text{otherwise,} \end{cases}$$

$$T_{u,k+1}^{2c} = \begin{cases} \inf\left\{s \geq T_{d,k}^{2c} : X_s - \inf_{t \in \left[T_{d,k}^{2c}; s\right]} X_t > 2c\right\} & \text{if } T_{d,k}^{2c} < +\infty, \\ +\infty & \text{otherwise.} \end{cases}$$

Remark 1. *Note that for any $s > 0$ there exists such $K < \infty$ that $T_{u,K}^{2c} > s$ or $T_{d,K}^{2c} > s$. Otherwise we would obtain two infinite sequences $(s(k))_{k=1}^{\infty}, (S(k))_{k=1}^{\infty}$ such that $0 \leq s(1) < S(1) < s(2) < S(2) < ... \leq s$ and $X_{S(k)} - X_{s(k)} \geq c$. But this is a contradiction since X is a càdlàg process and for any sequence such that $0 \leq s(1) < S(1) < s(2) < S(2) < ... \leq s$ sequences $\left(X_{S(k)}\right)_{k=1}^{\infty}, \left(X_{s(k)}\right)_{k=1}^{\infty}$ have a common limit.*

Now we define, for the given process X, the process X^c with the formulas

$$X_s^c = \begin{cases} X_0 & \text{if } s \in \left[0; T_{u,0}^{2c}\right); \\ \sup_{t \in \left[T_{u,k}^{2c}; s\right]} X_t - c & \text{if } s \in \left[T_{u,k}^{2c}; T_{d,k}^{2c}\right), k = 0, 1, 2, ...; \\ \inf_{t \in \left[T_{d,k}^{2c}; s\right]} X_t + c & \text{if } s \in \left[T_{d,k}^{2c}; T_{u,k+1}^{2c}\right), k = 0, 1, 2, ... \end{cases} \tag{1}$$

Remark 2. *Note that due to Remark 1, s belongs to one of the intervals $\left[0; T_{u,0}^{2c}\right), \left[T_{u,k}^{2c}; T_{d,k}^{2c}\right)$ or $\left[T_{d,k}^{2c}; T_{u,k+1}^{c}\right)$ for some $k = 0, 1, 2, ...$ and the process X_s^c is defined for every $s \geq 0$.*

Now we are to prove that X^c satisfies conditions (1)-(5).

Proof. (1) The process X^c has finite total on compact intervals, since it is monotonic on intervals of the form $\left[T_{u,k}^{2c}; T_{d,k}^{2c}\right), \left[T_{d,k}^{2c}; T_{u,k+1}^{c}\right)$ which sum up to the whole half-line $[0; +\infty)$.

(2) From formula (1) it follows that X^c is also càdlàg.

(3) In order to prove condition (3) we consider 3 possibilities.

- $s \in \left[0; T^{2c}_{u,0}\right)$. In this case, since $0 \le s < T^{2c}_u X \le T^{2c}_d X$, by definition of $T^{2c}_u X$ and $T^{2c}_d X$,

$$X_s - X^c_s = X_s - X_0 \in [-c; c].$$

- $s \in \left[T^{2c}_{u,k}; T^{2c}_{d,k}\right)$, for some $k = 0, 1, 2, \ldots$ In this case, by definition of $T^{2c}_{d,k}$, $\sup_{t\in\left[T^{2c}_{u,k};s\right]} X_t - X_s$ belongs to the interval $[0; 2c]$, hence

$$X_s - X^c_s = X_s - \sup_{t\in\left[T^{2c}_{u,k};s\right]} X_t + c \in [-c; c].$$

- $s \in \left[T^{2c}_{d,k}; T^{2c}_{u,k+1}\right)$ for some $k = 0, 1, 2, \ldots$ In this case $X_s - \inf_{t\in\left[T^{2c}_{d,k};s\right]} X_t$ belongs to the interval $[0; 2c]$, hence

$$X_s - X^c_s = X_s - \inf_{t\in\left[T^{2c}_{d,k};s\right]} X_t - c \in [-c; c].$$

(4) We will prove stronger fact than (4), namely that for every $s > 0$,

$$|\Delta X^c_s| \le |\Delta X_s|. \tag{2}$$

Indeed, from formula (1) it follows that for any $s \notin \left\{T^{2c}_{u,k}; T^{2c}_{d,k}\right\}$, (2) holds, hence let us assume that $s \in \left\{T^{2c}_{u,k}; T^{2c}_{d,k}\right\}$. We consider several possibilities. If $s = T^{2c}_{u,0}$ then, by the definition of $T^{2c}_{u,0}$,

$$X^c_s - X^c_{s-} = X_s - c - X_0 \ge 0 \text{ and } X^c_s - X^c_{s-} = X_s - X_0 - c \le X_s - X_{s-}.$$

If $s = T^{2c}_{u,k}, k = 1, 2, \ldots$, then, by the definition of $T^{2c}_{u,k}$,

$$X^c_s - X^c_{s-} = X_s - c - \left(\inf_{t\in\left[T^{2c}_{d,k-1};s\right]} X_t + c\right) = X_s - \inf_{t\in\left[T^{2c}_{d,k-1};s\right]} X_t - 2c \ge 0$$

and, on the other hand,

$$X_s^c - X_{s-}^c = X_s - \inf_{t \in \left[T_{d,k-1}^{2c}; s\right]} X_t - 2c \le X_s - X_{s-}.$$

Similar arguments may be applied for $s = T_{d,k}^{2c}, k = 0, 1, ...$

(5) The process X^c is adapted to the filtration F since it is adapted to any right continuous filtration containing the natural filtration of the process X.

□

Remark 3. *It is possible to define the process X^c in many different ways. For example, defining*

$$X^c = X_0 + UTV^c(X, \cdot) - DTV^c(X, \cdot)$$

we obtain a process satisfying all conditions (1)-(5) and having (on the intervals of the form $[0; T]$, $T > 0$) the smallest possible total variation among all processes, increments of which differ from the increments of the process X by no more than c. $UTV^c(X, \cdot)$ and DTV^c denote here upward and downward truncated variation processes. For more on upward truncated variation and downward truncated variation see e.g. [3], [4].

3. Pathwise Lebesgue-Stieltjes integration with respect to the processes from the class $\mathcal{X}^c$

Let us now consider a measurable space $(\Omega, \mathcal{F})$ equipped with a right-continuous filtration F and two processes X and Y with càdlàg paths, adapted to F. For $T > 0$ and for a sequence of processes $(X^c)_{c>0}$ with $X^c \in \mathcal{X}^c$ let us consider the sequence

$$\int_0^T Y_- \mathrm{d}X^c. \tag{3}$$

The integral in (3) is understood in the pathwise, Lebesgue-Stieltjes sense (recall that for any $c > 0$, X^c has bounded variation). We have

Theorem 4. *Assume that $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$ such that X and Y are semimartingales with respect to this measure and filtration F, which is complete under $\mathbb{P}$, then*

$$\int_0^T Y_- \mathrm{d}X^c \to^{ucp\mathbb{P}} \int_0^T Y_- \mathrm{d}X + \left[X^{cont}, Y^{cont}\right]_T \text{ as } c \downarrow 0,$$

where "$\to^{ucp\mathbb{P}}$" denotes uniform convergence on compacts in probability $\mathbb{P}$ and $[X^{cont}, Y^{cont}]_T$ denotes quadratic covariation of continuous parts X^{cont}, Y^{cont} of X and Y respectively.

Proof. Fixing $c > 0$ and using integration by parts formula (cf. [2, formula (1), page 519]) we get

$$Y_T X_T^c - Y_0 X_0^c = \int_0^T Y_{t-} \mathrm{d}X_t^c + \int_0^T X_{t-}^c \mathrm{d}Y_t + [Y, X^c]_T$$

(the above equality and subsequent equalities in the proof hold $\mathbb{P}$ a.s.). By the uniform convergence on $[0; T]$, $X_t^c \rightrightarrows X_t$ as $c \downarrow 0$ (note that the bound $|X^c| \leq |X| + K_T c$ and a.s. pointwise convergence $X_t^c \to X_t$ as $c \downarrow 0$ are sufficient) we get

$$\int_0^T X_{t-}^c \mathrm{d}Y_t \to^{ucp\mathbb{P}} \int_0^T X_{t-} \mathrm{d}Y_t.$$

Since X^c has locally finite variation, we have (cf. [2, Theorem 26.6 (viii)]),

$$[Y, X^c]_T = \sum_{0<s\leq T} \Delta Y_s \Delta X_s^c.$$

We calculate the (pathwise) limit

$$\lim_{c\downarrow 0} [Y, X^c]_T = \lim_{c\downarrow 0} \sum_{0<s\leq T} \Delta Y_s \Delta X_s^c = \sum_{0<s\leq T} \Delta Y_s \Delta X_s$$

(notice that for any $0 \leq s \leq T$, $|\Delta X_s^c| \leq L_T |\Delta X_s|$, thus the above sum is convergent by dominated convergence) and finally obtain

$$\begin{aligned}\int_0^T Y_{t-} \mathrm{d}X_t^c &= \left\{ Y_T X_T^c - Y_0 X_0^c - \int_0^T X_{t-}^c \mathrm{d}Y_t - [Y, X^c]_T \right\} \\ &\to^{ucp\mathbb{P}} Y_T X_T - Y_0 X_0 - \int_0^T X_{t-} \mathrm{d}Y_t - \sum_{0<s\leq T} \Delta Y_s \Delta X_s \text{ as } c \downarrow 0.\end{aligned} \tag{4}$$

On the other hand, again by the integration by parts formula, we obtain

$$\int_0^T X_{t-} \mathrm{d}Y_t = Y_T X_T - Y_0 X_0 - \int_0^T Y_{t-} \mathrm{d}X_t - [Y, X]_T . \tag{5}$$

Finally, comparing (4) and (5), and using [2, Corollary 26.15], we obtain

$$\begin{aligned}\int_0^T Y_{t-} \mathrm{d}X_t^c &\to^{ucp\mathbb{P}} \int_0^T Y_{t-} \mathrm{d}X_t + [Y, X]_T - \sum_{0<s\leq T} \Delta Y_s \Delta X_s \quad \text{as } c \downarrow 0 \\ &= \int_0^T Y_{t-} \mathrm{d}X_t + \left[X^{cont}, Y^{cont}\right]_T .\end{aligned}$$

□

Remark 5. *Assuming the existence of Mokobodzki's medial limits (cf. [6]), which one can not prove under standard Zermelo–Fraenkel set theory with the axiom of choice, Theorem 4 may be used to construct a universal process which coincides with stochastic integral, with the correction term* $[X^{cont}, Y^{cont}]_T$, *for a family of probability measures simultaneously. More precisely, we consider a family of probability measures* $\mathcal{P}$ *on* $(\Omega, \mathcal{F})$ *such that for each* $\mathbb{P} \in \mathcal{P}$ *the filtration* F *is complete under* $\mathbb{P}$ *and* X *and* Y *are semimartingales on the filtered probability space* $(\Omega, \mathcal{F}, F, \mathbb{P})$. *Considering any sequence*

$$\int_0^T Y_{-} \mathrm{d}X^{c(n)}, \;\; n = 1, 2, ...$$

with $c(n) \downarrow 0$, and using Theorem 4 and [7, Lemma 2.5] we obtain that there exists a universal, F adapted càdlàg process $I(Y, \mathrm{d}X, [0;\cdot])$, such that for all $\mathbb{P} \in \mathcal{P}$ and $T > 0$

$$I(Y, \mathrm{d}X, [0;T]) = \int_0^T Y_- \mathrm{d}X + \left[X^{cont}, Y^{cont}\right]_T \ \mathbb{P} \ a.s.$$

Note that to prove Theorem 4 we did not need the pathwise uniform convergence on compacts of the processes X^c to the process X; we might simply use local boundedness and a.s. pointwise convergence $X_t^c \to X_t$ as $c \downarrow 0$. Using the pathwise uniform convergence on compacts of the sequence $(X^c)_{c>0}$ we are able to prove a bit stronger result. We have

Theorem 6. *Assume that $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$ such that X and Y are semimartingales with respect to this measure and filtration F, which is complete under $\mathbb{P}$, then for any $T > 0$ and any sequence $(c(n))_{n\geq 1}$ such that $\sum_{n=1}^{\infty} c(n)^2 < +\infty$ we have*

$$\lim_{n\to+\infty} \sup_{0\leq t\leq T} \left| \int_0^t Y_- \mathrm{d}X^{c(n)} - \int_0^t Y_- \mathrm{d}X - \left[X^{cont}, Y^{cont}\right]_t \right| = 0 \ \mathbb{P} \ a.s.$$

Proof. Using integration by parts formula and the inequality $|X^c - X| \leq K_T c$, we estimate

$$\begin{aligned}
&\left| \int_0^t Y_- \mathrm{d}X^c - \int_0^t Y_- \mathrm{d}X - \left[X^{cont}, Y^{cont}\right]_t \right| \\
&= \left| Y_t \left(X_t^c - X_t\right) - Y_0 \left(X_0^c - X_0\right) - \sum_{0<s\leq t} \Delta Y_s \Delta \left(X_s^c - X_s\right) - \int_0^t \left(X_-^c - X\right) \mathrm{d}Y \right| \\
&\leq K_T c \left(|Y_0| + |Y_t|\right) + \left| \sum_{0<s\leq t} \Delta Y_s \Delta \left(X_s^c - X_s\right) \right| + \left| \int_0^t \left(X_-^c - X\right) \mathrm{d}Y \right|.
\end{aligned}$$

Thus we get

$$\sup_{0\le t\le T}\left|\int_0^t Y_- \mathrm{d}X^c - \int_0^t Y_- \mathrm{d}X - \left[X^{cont}, Y^{cont}\right]_t\right|$$
$$\le \quad K_T c\left(|Y_0| + \sup_{0\le t\le T}|Y_t|\right) + \sup_{0\le t\le T}\left|\sum_{0<s\le t}\Delta Y_s \Delta\left(X_s^c - X_s\right)\right| + \sup_{0\le t\le T}\left|\int_0^t \left(X_-^c - X\right)\mathrm{d}Y\right|.$$

Since Y has càdlàg paths, it is locally bounded and hence $K_T c\left(|Y_0| + \sup_{0\le t\le T}|Y_t|\right) \to 0$ $\mathbb{P}$ a.s. as $c \downarrow 0$.

Since for every $t \in [0;T]$, $|X_t^c - X_t| \le K_T c$ (condition (3)), for $s \in [0;t]$ we have $|\Delta\left(X_s^c - X_s\right)| \le 2K_T c$. Similarly, by condition (4),

$$|\Delta\left(X_s^c - X_s\right)| \le |\Delta X_s^c| + |\Delta X_s| \le (L_T + 1)|\Delta X_s|.$$

Thus we obtain that

$$|\Delta\left(X_s^c - X_s\right)| \le \min\left\{2K_T c, (L_T+1)|\Delta X_s|\right\} \le (2K_T + L_T + 1)\min\left\{c, |\Delta X_s|\right\}$$

and using this, we estimate

$$\sup_{0\le t\le T}\left|\sum_{0<s\le t}\Delta Y_s\left(\Delta X_s^c - \Delta X_s\right)\right| \le \sup_{0\le t\le T}\sqrt{\sum_{0<s\le t}|\Delta Y_s|^2}\sqrt{\sum_{0<s\le t}|\Delta\left(X_s^c - X_s\right)|^2}$$
$$= \sqrt{\sum_{0<s\le T}|\Delta Y_s|^2}\sqrt{\sum_{0<s\le T}|\Delta\left(X_s^c - X_s\right)|^2}$$
$$\le \sqrt{[\Delta Y]_T}\,(2K_T + L_T + 1)\sqrt{\sum_{0<s\le T}\min\left\{c^2, |\Delta X_s|^2\right\}} \to 0 \ \mathbb{P} \text{ a.s. as } c \downarrow 0.$$

In order to estimate

$$I_n(T) := \sup_{0\le t\le T}\left|\int_0^t \left(X_-^{c(n)} - X_-\right)\mathrm{d}Y\right|$$

let us decompose the semimartingale Y into a local martingale M with bounded jumps (hence a local L^2 martingale) and a process A with locally

finite variation (this is possible due to [2, Lemma 26.5] but the decomposition may depend on the measure $\mathbb{P}$), $Y = M + A$. Let $(\tau(k))_{k\geq 1}$ be a sequence of stopping times increasing to $+\infty$ such that $\left(M_{t\wedge\tau(k)}\right)_{t\geq 0}$ is a square integrable martingale. Using elementary estimate $(a+b)^2 \leq 2a^2 + 2b^2$ and the Burkholder inequality, on the set $\Omega_N = \{\omega \in \Omega : TV(A, [0;T]) \leq N\}$ we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau(k)}\left|\int_0^t \left(X_-^c - X_-\right) \mathrm{d}Y\right|^2; \Omega_N\right] \\
&\leq\ 2\mathbb{E}\sup_{0\leq t\leq T\wedge\tau(k)}\left|\int_0^t \left(X_-^c - X_-\right) \mathrm{d}M\right|^2 + 2\left[\mathbb{E}\left|\int_0^T \left|X_-^c - X_-\right| \mathrm{d}A\right|^2; \Omega_N\right] \\
&\leq\ 2\left(4K_T^2 c^2 \mathbb{E}\left[M,M\right]_{T\wedge\tau(k)} + K_T^2 c^2 N^2\right) \leq 8\left(\mathbb{E}\left[M,M\right]_{T\wedge\tau(k)} + N^2\right) K_T^2 c^2.
\end{aligned}
$$

Let now $(c(n))_{n\geq 1}$ be such a sequence that $\sum_{n=1}^{\infty} c(n)^2 < +\infty$. We have

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{n=1}^{\infty}\sup_{0\leq t\leq T\wedge\tau(k)}\left|\int_0^t \left(X_-^{c(n)} - X_-\right) \mathrm{d}Y\right|^2; \Omega_N\right] \\
&= \sum_{n=1}^{\infty}\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau(k)}\left|\int_0^t \left(X_-^{c(n)} - X_-\right) \mathrm{d}Y\right|^2; \Omega_N\right] \\
&\leq 8\left(\mathbb{E}\left[M,M\right]_{T\wedge\tau(k)} + N^2\right) K_T^2 \sum_{n=1}^{\infty} c(n)^2 < +\infty.
\end{aligned}
$$

Hence, the sequence $I_n(T\wedge\tau(k)), n = 1, 2, ...,$ converges to 0 on the set Ω_N. Since $\Omega = \bigcup_{N\geq 1}\Omega_N$, we get that $I_n(T\wedge\tau(k))$ converges $\mathbb{P}$ a.s. to 0. Finally, since $\tau(k) \to +\infty$ a.s. we get that $I_n(T)$ converges $\mathbb{P}$ a.s. to 0.

□

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